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Quine's Other Way Out

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Résumé : On montre que, avec la notion traditionnelle et grammaticale du prédicat comme ce qui reste de la phrase après l'enlèvement du sujet, le paradoxe de Russell, ou d'autres comparables comme le paradoxe de Grelling et le paradoxe de la prédication, ne posent aucun problème. L'interdit formel standard sur la substitution des prédicats impliquant des variables libres dans des schémas où ces variables deviendraient liées, suffit pour prévenir le développement des paradoxes standard. On discute ensuite des réarrangements requis dans les fondations de la Théorie des ensembles pour intégrer cette idée, et on explique les conséquences pour les questions étroitement liées de la Diagonalisation et du Théorème de Cantor.

Abstract: It is shown that, on the traditional, grammatical notion of a predicate as the remainder of a sentence once the subject term has been removed, there is no problem with Russell's Paradox, or comparable paradoxes such as Grelling's, and the Paradox of Predication. The standard formal ban on substituting predicates involving free variables into schemas where those variables would become bound is enough to prevent the standard paradoxes from developing. The re-arrangements required in the foundations of Set Theory to incorporate this insight are then discussed, and the consequences for the closely related matters Diagonalisation, and Cantor's Theorem explained.

1 Reflexive paradoxes

It has been pointed out in several places before [Slater 2004, 2005, 2007] that the Fregean tradition mixed up predicates with the forms of sentences. A predicate (in the old, and, outside of Logic books, still current sense) is a proper part of a sentence: it is that part of a sentence that remains after the subject is removed. Thus commonly, in English, the predicate is the latter part

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of a sentence, the part that follows the subject that commonly comes first. In this way the predicate in 'x is not a member of x' is 'is not a member of x', and the subject is the 'x' that has then been removed. On the other hand the form of the whole sentence is '(1) is not a member of (1)', and this has been thought of as a kind of 'predicate', following Frege. On this variant understanding of 'predicate' there is also a different understanding of 'subject'. A subject in this alternative sense is not what is maybe at the start of a sentence, but becomes a term or expression that may recur throughout the sentence. Thus if '(1) is not a member of (1)' is taken as the 'predicate' in 'x is not a member of x', then 'x' becomes the 'subject' in this second sense, because it replaces '(1)' at all occurrences, not just at the start.

The distinction enables us to see that something different is said of a and of b when, for example, we say of each that he shaves himself. For what is then predicated of each does not have the verbal form '(1) shaves (1)', but simply 'shaves himself', and the 'himself' has a variable referent, dependent on its contextual antecedent. So different properties are attributed to a and to b: the property of shaving a in the one case, and the property of shaving b in the other. Of course, all those who shave themselves might still contingently share a further property, and so form a set of those who have that property, and who, incidentally, are all of those who shave themselves, as when they are all together in a room: $(x)(Rx \equiv Sxx)$. Something of this form, of course, is always available in a finite universe, since a disjunctive list of predicates of the form 'is S to x' with variable 'x' can be provided. But there is no necessity that there is such an 'R' for all 'S', i.e., there is no logical equivalent of 'Sxx' of the form 'Rx' in general. Thus in an infinite universe, where a general description must be supplied rather than a list of cases, there is no constant predicate available in place of the variable 'is S to x'. The natural language form 'is S to itself' certainly covers all cases, but it also contains a variable in the shape of the pronoun 'itself' which gains its referent from the subject the predicate is attached to.

The point resolves a number of puzzles that have bedevilled twentieth century logic. For, in connection with Grelling's Paradox, a problem arises when we use such a word as 'heterological' for what 'x' is when 'x' does not apply to 'x'. For then the variable within the (old-style) predicate 'does not apply to 'x'' is obscured, since such words are properly used only for constant predicates. If instead we use 'not self applicable', the variable nature of the predicate is more apparent, although we still might forget that substituting 'not self applicable' for 'x' in:

> 'x' is not self applicable iff 'x' does not apply to 'x',

means substituting it for 'self' as well as 'x', since there are four references to 'x' in the statement, and not just three. Substituting 'not self applicable' ('NSA') for 'x' in this statement does not lead to

> 'NSA' is NSA iff 'NSA' does not apply to 'NSA',

but to

'NSA' is not 'NSA' applicable iff 'NSA' does not apply to 'NSA',

which is unexceptionable.

The same applies to Russell's Paradox, the Paradox of Predication, and other forms of Grelling's Paradox. For, notoriously, if we try to represent 'x is not a member of itself' as 'x is a member of R' for some fixed 'R', then a contradiction ensues. But none does if we respect the variable nature of 'itself'. What x is necessarily a member of, for instance, if it is not a member of itself, is its complement. But 'its complement' contains the contextual element 'its', and so in

x is a member of its complement if and only if x is not a member of x,

($x \in x' \equiv x \notin x$) substitution of 'its complement' ('IC') for 'x' leads not to the contradictory

IC is a member of IC if and only if IC is not a member of IC,

($x' \in x' \equiv x' \notin x'$) but to the unexceptionable

IC is a member of IC's complement if and only if IC is not a member of IC,

($x' \in x \equiv x' \notin x'$) once one remembers that there is a variable item in the predicate 'is a member of its complement'. In the Paradox of Predication the concern is with 'x is a property it does not possess', or 'x is a property but does not possess that property', i.e., '$(\exists P)(x = P \& \neg(x \text{has} P))$'. But this is '$x = P^* \& \neg(xhasP^*)$' with $P^* = \varepsilon P(x = P \& \neg(xhasP))$ in the epsilon reduction [Leisenring 1969, 19], [Slater 2006, sections 8 and 9], which clearly shows that the property attributed to x in the (old-style) predicate is not constant, but varies with x. Likewise with Grelling's Paradox in the form "x' does not possess the property it expresses', or "x' expresses but does not possess a certain property', i.e., '$(\exists P)$('x' expresses$P \& \neg$('$x'hasP$))'.

2 Being free for x

However, it has recently come to my attention that there is another way of obtaining this conclusion using a standard feature of formal logic. For if the substituted 'F' in the naïve abstraction scheme

$$(\exists y)(x)(x \text{ is a member of } y \equiv Fx),$$

had to be a predicate in the old style, then the substitution of 'is not a member of x' for 'F' would violate a formal restriction. If one tried to derive Russell's

Paradox from this abstraction scheme by substituting the predicate 'is not a member of x' for 'F', to get 'x is not a member of x' for 'Fx', then this would violate the restriction that variables free in a predicate must not be such as to be captured by quantifiers in the scheme into which the predicate is substituted [Quine 1959, 141]. For the variable 'x' in 'is not a member of x' would become bound by the quantifier '(x)'.

There is no problem with introducing occurrences of other variables in the substituted predicate, but there is a quite general problem with bringing in a variable free in the substituted predicate that would be bound in the scheme it is substituted into. In an example from Quine, consider the substitution of 'Gx' for 'F' in

$$Fy \supset (\exists x)Fx.$$

This implication is formally valid, so the given substitution is improper since it would yield

$$Gxy \supset (\exists x)Gxx,$$

which is invalid [Quine 1959, 144].

Quine himself overlooked the way this point provides a way out from Russell's Paradox. That was no doubt because the novel Fregean grammar was burnt well into him. In the way Fregeans think of it, it is quite proper that, in the scheme of naive abstraction, '$F(1)$' be replaced by '(1) is not a member of (1)', to yield

$$(\exists y)(x)(x \text{ is a member of } y \equiv x \text{ is not a member of } x).$$

Putting it this way, one is using Quine's device of 'placeholders' to indicate the argument-places of '$F(1)$'. The point to note is that the complex 'predicate' (strictly 'form of a sentence') that then replaces '$F(1)$' does not contain any occurrences of 'x', hence the above bar on capturing seemingly does not apply. Fregeans would think of themselves as substituting '(1) is not a member of (1)' not for 'Fx' but for '$F(1)$', where the argument places marked by '(1)' are filled by whatever fills the argument place of '$F(1)$'—in the above case 'x'.

But if we keep to the traditional notion of predicate as the remainder of a sentence after the removal of (in English) the first occurrence of its subject, then clearly Quine's restriction will enable us to escape the paradox that results from the Fregean way of looking at the matter. More exactly, it will enable us to escape from paradox with any substitution into the abstraction scheme

$$(\exists y)(x)(x \text{ is a member of } y \equiv Fx),$$

that does not violate the above bar on capturing. For the further point that needs to be made is that that does not preclude having further abstraction schemes applying when there is reflexivity in the predicate. There is no problem with replacing the 'F' above with any constant, old style predicate, or even such a predicate involving another variable, like 'Rz'. But being unable to replace the above 'F' with 'Rx' leaves us with the need for an abstraction scheme applicable when 'Rxx' is on the right hand side. That is no problem, however, since the way to handle relations quite generally, and so equally when the subject is repeated, is to bring in sets of ordered pairs.

If a is shaving a, then, as before, a has the property of shaving a (also the property of being shaved by a). But a also stands in a relation to himself: he and himself form a shaving (i.e., shaver-shaved) pair. Moreover, if a is shaving a, and b is shaving b, then the *same* relation is involved—a relation that a holds with himself, and b holds with *himself* (sic, notice the change of referent); and that relation is not a specifically 'reflexive relation', since it is the same relation that a would have with b, if a were shaving b. Thus quite generally,

$$(\exists y)(x)(z)(< z, x > \text{ is a member of } y \equiv Rzx),$$

and the same y is involved if $z = x$, even though y is not just a set of ordered pairs whose members, in each case, are the same. But, in the particular case

$$(\exists y)(x)(< x, x > \text{ is a member of } y \equiv x \text{ is not a member of } x),$$

we only get, on substitution,

$$< y, y > \text{is a member of } y \equiv y \text{ is not a member of } y,$$

which is not a contradiction.

Surprisingly, therefore, we must conclude that, from a traditional perspective, Frege got into his problem with Russell's Paradox through forgetting the applicability of the elementary notion of 'being free for' to the case.

3 Axioms for set theory

Of course the elementary, and rather banal principles above are the basis for more interesting and complicated results, since once sets for elementary predicates are defined, those for non-elementary predicates can be constructed out of them by standard set-theoretic processes.

Thus, for a start, from the Abstraction Axiom (given Extensionality, which ensures uniqueness of referent for the epsilon term) we can invariably write '$\{x : Px\}$' or '$\varepsilon z(y)(y \in z \equiv Py)$' for the set of Ps (where the (old-style predicate) 'P' in 'Py' contains no occurrence of 'y'). Repeated variables in a relation must be handled differently, as above, but since 'x is P and x is Q' is the same as 'x is P and Q' ('x is R to y, and x is S to y' is the same as 'x is R and S to y' etc.), the repeated variables in finite conjunctions like '$Px\&\ Qx$' can be handled using the normal definition of set intersection. Thus:

$$\{x : Px \ \&\ Qx\} = \{x : x \in (\{y : Py\} \cap \{y : Qy\})\}.$$

Likewise with the union of two sets, and the complement of a set:

$$\{x : Px\ v\ Qx\} = \{x : x \in (\{y : Py\} \cup \{y : Qy\})\}; \{x : \neg Px\} = \{x : x \notin \{y : Py\}\}.$$

The null set can then be defined as the intersection of $\{x : Fx\}$ and $\{x : \neg Fx\}$ (for any 'F'), i.e.,

$$\varnothing = \varepsilon y(x)(x \in y \equiv Fx) \cap \varepsilon y(x)(x \in y \equiv \neg Fx),$$

and the universal set likewise as the union of $\{x : Fx\}$ and $\{x : \neg Fx\}$ (for any 'F').

As for the standard axioms of Set Theory, the present approach has the advantage of making most of them redundant. Thus the Axiom of Regularity is not required since there is nothing suspect about expressions like '$x \in x$', and their more complex kin. Some of the functions of the Axiom of Choice are taken over by the properties of epsilon terms (as in Bernays' formulation of Set Theory, [Bernays 1968]). For

$$(\exists x)(x \in y) \equiv \varepsilon x(x \in y) \in y,$$

and so the appropriate epsilon term always provides a selection from a non-empty set. The Power Set Axiom follows using Abstraction on the definition of a subset, for given

$$y \subset x \equiv (z)(z \in y \supset z \in x),$$

and

$$(\exists z)(t)(t \in z \equiv t \subset x),$$

then one can always define

$$\Psi x = \{y : y \subset x\}(= \varepsilon z(t)(t \in z \equiv t \subset x)).$$

The Axiom of Pairs is now an immediate inference from Abstraction and the definition of the union of two sets, since

$$(\exists y)(x)(x \in y \equiv x = z),$$

yields

$$x = z \equiv x \in \{t : t = z\},$$

and so it follows, using the process of (finite) set union above, that

$$(\exists y)(x)(x \in y \equiv x = z \ v \ x \ = t).$$

The Axiom of Separation in the form

$$(\exists y)(x)(x \in y \equiv (x \in z \ \& \ Px)),$$

(where 'Px' is as before) is now an immediate inference from Abstraction and the definition of the intersection of two sets. But the Axiom of Separation is standardly expressed using, in place of 'Px', a formula in which 'x' might occur any number of times. So that will not follow in the present case, without a series of further assumptions like

$$(y)(\exists x)(t)(t \in x \equiv < t, t > \in y).$$

This (and its kin with larger ordered sets) clearly holds if the set corresponding to 'y' is finite, and so can be listed and not just known descriptively, i.e., 'intensionally'. But it cannot hold in general, since it is just this kind of assumption, we now see, that generates Russell's Paradox.

So care must be taken with, for instance, such equivalences as

$$Rss \equiv (\exists t)(s = t \ \& \ Rtt).$$

The R.H.S. here looks like it might be of the required constant form 'Ps', and so the further assumption above may seem to be automatically satisfied. Thus 's shaves himself' is equivalent to 's is someone who shaves himself' and the predicate 'is someone who shaves himself' might seem to have a constant sense. The subject-predicate structure of the R.H.S., however, is more fully displayed in its epsilon equivalent:

$$s = t^* \ \& \ Rt^*t^*,$$

where $t^* = \varepsilon t(s = t \ \& \ Rtt)$. So in the old-style predicate in question (i.e., the portion of this last expression after the initial 's') there are again further occurrences of the subject, making the referent of the pronoun 'someone' in 'is someone who shaves himself' not constant, but a function of the subject the predicate is applied to. So while there is a constant syntactic predicate, the epsilon analysis reveals it expresses a variable property, as with 'shaves himself', 'does not apply to itself', etc.

What the above equivalence does logically ensure is that something of the following form is provable:

$$(t)(< t, t > \in x \equiv < t, t, t, t > \in y).$$

But with 'Rss' as 's shaves himself' then, as before, only contingently (and thus only with a finite set being involved) could there be a z such that

$$(t)(t \in z \equiv < t, t > \in x).$$

So 'Separation' in its traditional form is not automatically guaranteed, and that also means that the Axiom of Choice, on which the full form of Separation is clearly based must not be assumed in general. For one moves in the further assumptions above from a set of ordered sets with iterated members to a set that selects just one member from each of those ordered sets (the problem being not in the selection of the various members, but in whether there is a set of all those selected when it would have to be given descriptively, or 'intensionally', being infinite). That leaves Abstraction, and Extensionality as the only two set-theoretic principles that are totally justified.

4 Further matters

There are consequences for the understanding of Diagonalisation, of course, with which Russell's Paradox is closely related. For what has been called 'Cantor's Theorem' seems to show that the power set of any set has greater cardinality than the set itself. If 'x' ranges over members of a set, and 'S_x' over correlated subsets of the set, then Cantor argued that

$$(x)(x \in S_y \equiv x \notin S_x),$$

must define a further subset, S_y, i.e., the 'y' cannot name a member of the set. But if that was so then it would follow that there could be no universal set. For each of its subsets would have to be a member of it, being sets, making the cardinality of the universal set at least as great as the cardinality of its power set—a contradiction. But a universal set is easily defined, as before. So there must be something wrong with Cantor's argument, and what is wrong is now easy to diagnose. For what is true, for a start, is merely that

$$(\exists z)(x)(< x, x > \in z \equiv x \notin S_x),$$

so Cantor needed a further premise

$$(\exists y)(t)(t \in y \equiv < t, t > \in \varepsilon z(x)(< x, x > \in z \equiv x \notin S_x)),$$

to establish that his 'theorem' held in general.

Likewise with other forms of 'Cantor's Theorem'. For given a defined sequence of functions of one variable, $f_x(y)$, onto (0,1) then

$$F_r(x) = 1 - f_x(x)$$

will define a different function, i.e., the 'r' will not be one of the 'x's. But in the extreme case, where the sequence contains *all* functions of one variable onto (0,1), evidently no new function can be defined in this way, without contradiction. So it is not just that there might be something like a 'non-recursive function' in such a case beyond recursive ones. What there is in the extreme case is a sequence of functions without *any* definable function of one variable generating $f_x(x)$, because from the index 'x' there is no definable function generating the function with that index, and so no 'F' such that $f_x(x) = F(x)$.

The situation, in other words [Slater 2000, 94–95], parallels that for computable functions. For while all computable functions of one numerical variable onto (0,1) are enumerable, there is no way to specifically enumerate just those that have completely defined values (i.e., which are not just partial but total functions), otherwise the halting problem would be solved. Hence the ordinal numbers of those functions that are total, although denumerable, are not enumerable. There is, in other words, a further kind of expression, which is like that for a binary 'decimal' except certain places are undefined. These expressions are enumerable, but diagonalisation does not produce a further one of them, since neither $f_m(n)$, nor $1 - f_n(n)$ need equal anything. Amongst the functions which generate these expressions are all the total functions of one variable, but we cannot, in general, determine which these functions are. Even if $f_m(n)$ is total, which function it is is only determinable from its ordinal place amongst all the computable functions of one variable, not from its ordinal place amongst the total functions of this sort, with the result that, if the latter is 'm', then $f_m(n)$ is not a calculable function of m. Of course, if one *specifies* a sequence just of total functions that makes it the case that which function is the mth in that sequence is determinable from m, and $1 - f_n(n)$ will then be a further, distinct total function of n. But it is only the specification of such a sequence which makes $f_m(n)$ a function both of m and of n, and so there is no further diagonal function in an unspecified case, much as there was no diagonal set in the extreme case before.

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